

# A Brane World in an Arbitrary Number of Dimensions without $Z_2$ Symmetry

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We consider a brane world in an arbitrary number of dimensions without  $Z_2$  symmetry and derive the effective Einstein equation on the brane, where its right-hand side is given by the matter on the brane and the curvature in the bulk. This is achieved by first deriving the junction conditions for a non- $Z_2$  symmetric brane and second solving the Gauss equation, which relates the mean extrinsic curvature of the brane to the curvature in the bulk, with respect to the mean extrinsic curvature. The latter corresponds to formulating an explicit junction condition on the mean of the extrinsic curvature, analogue to the Israel junction condition for the jump of the extrinsic curvature. We find that there appears a new type of effective anisotropic fluid on the right-hand side of the effective Einstein equation due to the fact that there is no  $Z_2$  symmetry. The derived equation is a basic equation for the study of Kaluza-Klein brane worlds in which some dimensions on the brane are compactified or for a regularization scheme for a higher codimension brane world, where the Kaluza-Klein compactification on the brane is regarded as a means to regularize the uncontrollable spacetime singularity created by the higher codimension brane.

## §1. Introduction

String theory suggests that our universe is not four dimensional but, rather, a submanifold (brane) embedded in a higher-dimensional spacetime (bulk). In particular, Randall and Sundrum (RS)<sup>1),2)</sup> proposed an interesting brane world model inspired by the Horava-Witten model.<sup>3)</sup> The RS model assumes a codimension-1 brane with  $Z_2$  symmetry embedded in the bulk with a negative cosmological constant. However, to reconcile a higher-dimensional theory with the observed four-dimensional spacetime, the RS model is not sufficient. Since string theory suggests that the number of bulk dimensions is 10 or 11, the corresponding number of codimensions is 6 or 7. Therefore, we must consider a higher-codimension brane world. But a higher-codimension brane world has the serious problem that the brane becomes an uncontrollable spacetime singularity due to its self-gravity, except possibly for a codimension-2 brane world, which may give a reasonable cosmology.<sup>4)</sup> Thus it is necessary to develop a regularization method to realize a reasonable higher-codimension brane world.

For the above-stated purpose, we focus on a specific regularization scheme, which we now describe. Let us consider a codimension- $(q + 1)$  brane in an  $n$ -dimensional spacetime. We regularize this brane by expanding it into  $q$ -dimensions, so that it becomes a codimension-1 brane with  $q$  compact dimensions on the brane. In 3 dimensions, for example, this corresponds to replacing a point by a small 2-sphere.

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Note that the resulting codimension-1 brane will not have the  $Z_2$  symmetry.

This regularization scheme is essentially the same as the Kaluza-Klein (KK) compactification of  $q$  spatial dimensions on the brane, which is called the KK brane world.<sup>5)</sup> In the codimension-2 case, Peloso et al.<sup>6)</sup> considered KK brane worlds with the internal two-dimensional space compactified by a flux (see also Refs. 7)–9)). They showed using a perturbative analysis that four-dimensional Einstein gravity is recovered on the brane. Recently, Kanno et al.<sup>10)</sup> investigated the cosmology of the KK brane worlds in a non-trivial bulk background.

In this paper, partly to give a framework for the KK brane worlds and partly as a first step to formulate the above-mentioned regularization scheme for brane worlds of arbitrary codimension, we consider a codimension-1 brane world in an arbitrary number of spacetime dimensions without  $Z_2$  symmetry and derive an effective Einstein equation on the brane, which is a generalization of the effective Einstein equation on the brane with  $Z_2$  symmetry derived by Shiromizu, Maeda and Sasaki.<sup>11)</sup>

There has been some work on non- $Z_2$  symmetric branes.<sup>12)–16)</sup> The work most relevant to the present one is that of Battye et al.,<sup>12)</sup> in which the non- $Z_2$  symmetric brane world is investigated. They study the junction condition in detail and point out that the effective Einstein equation has terms involving the mean of the extrinsic curvature across the brane which are not explicitly expressed in terms of either the matter on the brane or the curvature in the bulk. Then, they focused their investigation on a spatially homogeneous, isotropic brane. Our purpose here is to solve this problem and express the effective Einstein equation solely in terms of the matter on the brane and the curvature in the bulk, and also to present a straightforward generalization in which the number of spacetime dimensions of the bulk is extended from 5 to  $n$ .

This paper is organized as follows. In § 2, we review the basic equations, in particular we give the junction condition for non- $Z_2$  symmetric branes in Eq. (2.19), first obtained by Battye et al.<sup>12)</sup> in five dimensions and generalized to arbitrary dimensions by Chatillon et al.<sup>5)</sup> We present the effective Einstein equation, but with terms which still involve the mean of the extrinsic curvature, which we call the pre-effective Einstein equation. In § 3, using the Gauss equation, we solve for the mean of the extrinsic curvature across the brane in terms of the matter on the brane and the curvature in the bulk. § 4 is devoted to a conclusion and discussion.

Throughout the paper, we use square brackets to denote the jump of a quantity across the brane and angled brackets to denote its mean. For an arbitrary tensor  $\mathcal{A}$  (with tensor indices suppressed), we define

$$[\mathcal{A}] \equiv \mathcal{A}^+ - \mathcal{A}^-, \quad (1.1)$$

$$\langle \mathcal{A} \rangle \equiv \frac{1}{2}(\mathcal{A}^+ + \mathcal{A}^-), \quad (1.2)$$

where the superscript  $+$  denotes the side of the brane from which the normal vector  $n^A$  (to be introduced in § 2) points toward the bulk. We note that the brackets  $\langle \mathcal{AB} \rangle$  and  $[\mathcal{AB}]$  for the products of two tensors  $\mathcal{A}$  and  $\mathcal{B}$  can be expressed as

$$\langle \mathcal{AB} \rangle = \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle + \frac{1}{4}[\mathcal{A}][\mathcal{B}], \quad (1.3)$$

$$[\mathcal{AB}] = \langle \mathcal{A} \rangle [\mathcal{B}] + [\mathcal{A}] \langle \mathcal{B} \rangle. \quad (1.4)$$

## §2. Junction condition for $[K_{ab}]$

### 2.1. $(n-1)+1$ decomposition

We consider a family of  $(n-1)$ -dimensional timelike hypersurfaces (slicing) in an  $n$ -dimensional spacetime and identify one of them as a brane (i.e., a singular hypersurface). We denote the bulk metric by  $g_{MN}$  with the coordinates  $x^M$ , where  $M = 0, 1, \dots, n-1$ . We denote the vector field unit normal to the hypersurfaces by  $n^M$ . Then the induced metric  $\gamma_{MN}$  on the hypersurfaces is given by

$$\gamma_{MN} = g_{MN} - n_M n_N. \quad (2.1)$$

We express the internal coordinates on the brane by  $y^a$ , with  $a = 0, 1, \dots, n-2$ . The brane is embedded in the bulk as  $x^M = x^M(y^a)$ . Then, the induced metric on the brane is expressed as

$$\begin{aligned} \gamma_{ab} &= g_{MN} \frac{\partial x^M}{\partial y^a} \frac{\partial x^N}{\partial y^b}, \\ \gamma^{AB} &= \gamma^{ab} \frac{\partial x^M}{\partial y^a} \frac{\partial x^N}{\partial y^b}. \end{aligned} \quad (2.2)$$

The metric  $\gamma^A_B = \gamma^{AC} g_{CB}$  acts as a projection operator, projecting bulk tensors onto the brane.

The extrinsic curvature  $K_{ab}$  of the brane is defined as

$$K_{ab} \equiv \frac{1}{2} \mathcal{L}_n \gamma_{ab} = \gamma_b^M \gamma_a^N \nabla_N n_M. \quad (2.3)$$

Denoting the  $n$ -dimensional Riemann tensor by  $\mathcal{R}_{ABCD}$  and the  $(n-1)$ -dimensional Riemann tensor by  $\bar{R}_{abcd}$ , the Gauss-Codazzi equations are expressed as

$$\begin{aligned} \mathcal{R}_{ABCD} \gamma_a^A \gamma_b^B \gamma_c^C \gamma_d^D &= \bar{R}_{abcd} + 2K_{a[d} K_{c]b}, \\ \mathcal{R}_{MABC} n^M \gamma_a^A \gamma_b^B \gamma_c^C &= 2\bar{\nabla}_{[c} K_{b]a}, \end{aligned} \quad (2.4)$$

where  $\bar{\nabla}_a \equiv \gamma_a^M \nabla_M$  denotes the covariant derivative associated with the brane induced metric  $\gamma_{ab}$ . The bulk Riemann tensor is expressed in terms of the bulk Ricci tensor and Weyl curvature  $\mathcal{C}_{ABCD}$  as

$$\mathcal{R}_{AB}{}^{CD} = \mathcal{C}_{AB}{}^{CD} + \frac{4}{n-2} g_{[A}^{[C} \mathcal{R}_{B]}^{D]} - \frac{2}{(n-2)(n-1)} g_A^{[C} g_B^{D]} \mathcal{R}, \quad (2.5)$$

where  $\mathcal{R}_{AB} \equiv R^C_{ACB}$  and  $\mathcal{R} = \mathcal{R}^A_A$ . Using this, the Gauss equation gives

$$\begin{aligned} \bar{R}_{ab} &= \frac{n-3}{n-2} \mathcal{R}_{AB} \gamma_a^A \gamma_b^B + \frac{1}{n-2} \mathcal{R}_{CD} \gamma^{CD} \gamma_{ab} - \frac{1}{n-1} \mathcal{R} \gamma_{ab} \\ &\quad + K K_{ab} - K_a^c K_{bc} + \mathcal{E}_{ab}, \end{aligned} \quad (2.6)$$

where  $\bar{R}_{ab} = \bar{R}^c_{acb}$ , and  $\mathcal{E}_{ab}$  is the projected Weyl curvature on the brane, defined by

$$\mathcal{E}_{ab} \equiv \mathcal{C}_{ACBD} n^C n^D \gamma_a^A \gamma_b^B. \quad (2.7)$$

For convenience, we introduce the tensor  $\mathcal{F}_{ab}$  defined by

$$\mathcal{F}_{ab} \equiv \frac{n-3}{n-2} \mathcal{R}_{AB} \gamma_a^A \gamma_b^B + \frac{1}{n-2} \mathcal{R}_{CD} \gamma^{CD} \gamma_{ab} - \frac{1}{n-1} \mathcal{R} \gamma_{ab} + \mathcal{E}_{ab}. \quad (2.8)$$

We denote its trace and traceless parts by  $\mathcal{F}$  and  $\omega_{ab}$ ,

$$\mathcal{F} \equiv \mathcal{F}^a_a = 2\mathcal{R}_{AB} \gamma^{AB} - \mathcal{R} = 2\mathcal{G}_{MN} n^M n^N, \quad (2.9)$$

$$\omega_{ab} \equiv \frac{n-3}{n-2} (\mathcal{R}_{AB} \gamma_a^A \gamma_b^B - \frac{1}{n-1} \mathcal{R}_{AB} \gamma^{AB} \gamma_{ab}) + \mathcal{E}_{ab}, \quad (2.10)$$

where  $\mathcal{G}_{AB} = \mathcal{R}_{AB} - \frac{1}{2} \mathcal{R} g_{AB}$  is the  $n$ -dimensional Einstein tensor. Thus,  $\mathcal{F}_{ab}$  is decomposed as

$$\mathcal{F}_{ab} = \frac{\mathcal{F}}{n-1} \gamma_{ab} + \omega_{ab}. \quad (2.11)$$

Using  $\mathcal{F}_{ab}$ , we can rewrite the Gauss equation (2.6) as

$$\bar{R}_{ab} = \mathcal{F}_{ab} + K K_{ab} - K^c_a K_{bc}. \quad (2.12)$$

Using the fact that the brane induced metric satisfies the junction condition  $[\gamma_{ab}] = 0$ , the Gauss equation can be decomposed into two equations,

$$\begin{aligned} \langle \bar{R}_{ab} \rangle &= \bar{R}_{ab} = \langle \mathcal{F}_{ab} \rangle + \langle K K_{ab} \rangle - \langle K^c_a K_{bc} \rangle \\ &= \langle \mathcal{F}_{ab} \rangle + \frac{1}{4} ([K][K_{ab}] - [K^c_a][K_{bc}]) + \langle K \rangle \langle K_{ab} \rangle - \langle K^c_a \rangle \langle K_{bc} \rangle, \end{aligned} \quad (2.13)$$

$$\begin{aligned} [\bar{R}_{ab}] &= 0 = [\mathcal{F}_{ab}] + [K K_{ab}] - [K^c_a K_{bc}] \\ &= [\mathcal{F}_{ab}] + \langle K \rangle [K_{ab}] + [K] \langle K_{ab} \rangle - 2 \langle K^c_a \rangle [K_{bc}], \end{aligned} \quad (2.14)$$

where we have used the two product relations (1.3) and (1.4).

## 2.2. Israel junction condition

We implicitly assume Einstein gravity in the bulk,

$$\mathcal{G}_{AB} = \kappa_{(n)}^2 T_{AB}, \quad (2.15)$$

where  $\kappa_{(n)}^2$  is the  $n$ -dimensional gravitational coupling constant. Since we focus on the effective Einstein equation on the brane, it is unnecessary to specify the matter in the bulk and solve for the bulk geometry in the treatment below. We simply assume that the bulk geometry is defined consistently.

The Israel junction condition<sup>17)</sup> is

$$[K \gamma_{ab} - K_{ab}] = \kappa_{(n)}^2 \bar{T}_{ab}, \quad (2.16)$$

where  $\bar{T}_{ab}$  is the energy-momentum tensor of the matter on the brane. We decompose the energy-momentum tensor as

$$\bar{T}_{ab} = -\lambda\gamma_{ab} + \tau_{ab}, \quad (2.17)$$

where  $\lambda$  is the brane tension. Using this decomposition, we rewrite the junction condition as

$$[K_{ab}] = -\kappa_{(n)}^2 \left( \tau_{ab} + \frac{\lambda - \tau}{n-2} \gamma_{ab} \right). \quad (2.18)$$

Using this junction condition, the jump of the Gauss equation (2.14) is written

$$-\frac{1}{\kappa_{(n)}^2} [\mathcal{F}_{ab}] = 2\tau_{(a}^c \langle K_{b)c} \rangle - \frac{(n-3)\lambda - \tau}{n-2} \langle K_{ab} \rangle - \langle K \rangle \left( \tau_{ab} + \frac{\lambda - \tau}{n-2} \gamma_{ab} \right), \quad (2.19)$$

where  $\tau = \tau^a_a$ . We note that this is the key equation that relates the mean extrinsic curvature  $\langle K_{ab} \rangle$  to the bulk curvature and the brane energy-momentum tensor. As discussed below, a preliminary form of the effective Einstein equation on the brane, which we call the pre-effective Einstein equation on the brane, contains many terms involving  $\langle K_{ab} \rangle$ . In § 3, we solve this equation for the mean extrinsic curvature to obtain a meaningful effective Einstein equation on the brane.

In passing, we mention a necessary condition for the brane to be embedded in the bulk in a consistent manner.<sup>18)</sup> Taking the trace of Eq. (2.19) or Eq. (2.14) with the Israel junction condition (2.16), we obtain

$$[\mathcal{F}] = 2\kappa_{(n)}^2 \bar{T}^{ab} \langle K_{ab} \rangle. \quad (2.20)$$

Comparing this with Eq. (2.9), we obtain a part of the relation between the brane and the bulk,

$$2\kappa_{(n)}^2 \bar{T}^{ab} \langle K_{ab} \rangle = [\mathcal{G}_{MN}] n^M n^N. \quad (2.21)$$

We mention that Israel derived this equation in the case  $[\mathcal{F}] = 2[\mathcal{G}_{MN}] n^M n^N = 0$ .<sup>17)</sup>

### 2.3. Pre-effective Einstein equation on the brane

Using the results of the previous subsection, we construct the effective Einstein equation on the brane without any symmetry. For convenience, we also decompose the extrinsic curvature into its trace and traceless parts,

$$K_{ab} = \frac{K}{n-1} \gamma_{ab} + \sigma_{ab}, \quad (2.22)$$

where the traceless part,  $\sigma_{ab}$ , describes the shear of the normal vector  $n^M$ , and the trace,  $K$ , describes its expansion.

The effective Einstein equation is derived from the mean of the Gauss equation (2.13). Inserting the junction condition (2.18) into it, together with the decomposition formulas (2.11) and (2.22) for  $\mathcal{F}_{ab}$  and  $K_{ab}$ , respectively, we obtain

$$\bar{G}_{ab} = -\bar{\Lambda}\gamma_{ab} + \kappa_{(n-1)}^2 \tau_{ab} + \frac{\kappa_{(n-1)}^2}{\lambda} S_{ab} + \langle \Omega_{ab} \rangle - \langle \sigma_a^c \rangle \langle \sigma_{bc} \rangle, \quad (2.23)$$

where the effective  $(n-1)$ -dimensional gravitational constant is defined by

$$\kappa_{(n-1)}^2 = \frac{n-3}{4(n-2)} \kappa_{(n)}^4 \lambda, \quad (2.24)$$

and the other terms are given by

$$\bar{\Lambda} = \frac{n-3}{2(n-1)} \langle \mathcal{F} \rangle + \frac{1}{2} \kappa_{(n-1)}^2 \lambda + \frac{(n-2)(n-3)}{2(n-1)^2} \langle K \rangle^2 - \frac{1}{2} \langle \sigma^{cd} \rangle \langle \sigma_{cd} \rangle, \quad (2.25)$$

$$S_{ab} = \frac{\tau}{n-3} \tau_{ab} - \frac{\tau^2}{2(n-3)} \gamma_{ab} - \frac{n-2}{n-3} \tau_a^c \tau_{bc} + \frac{n-2}{2(n-3)} \tau^{cd} \tau_{cd} \gamma_{ab}, \quad (2.26)$$

$$\langle \Omega_{ab} \rangle = \langle \omega_{ab} \rangle + \frac{n-3}{n-1} \langle K \rangle \langle \sigma_{ab} \rangle. \quad (2.27)$$

The first term  $\bar{\Lambda}$  on the right-hand side of Eq. (2.23) represents the effective cosmological constant. We note, however, that this quantity may not be constant in general, as is clear from its expression in Eq. (2.25). It reduces to a constant, for example, in the  $Z_2$  symmetric case, in which all the angled brackets except for  $\mathcal{F}_{ab}$  are zero. The second and third terms are contributions from the energy-momentum tensor on the brane and its quadratic term, which are the same as in the  $Z_2$  symmetric case.<sup>11)</sup> The fourth traceless term,  $\langle \Omega_{ab} \rangle$ , which is traceless by definition, is an extension of the  $\mathcal{E}_{ab}$  term in the  $Z_2$  symmetric case. Finally, the last term is a new term, which has no analog in the  $Z_2$  symmetric case. As is clear from its definition, this term arises from the square of the mean extrinsic curvature  $\langle K_{ab} \rangle$ , and it vanishes only if the traceless part of  $\langle K_{ab} \rangle$  is zero.

The above effective Einstein equation, (2.23), is completely general in the sense that no symmetry has been imposed. However, it is useless, except in the  $Z_2$  symmetric case, because it depends strongly on the unknown mean extrinsic curvature  $\langle K_{ab} \rangle$ : It is contained in  $\bar{\Lambda}$ ,  $\langle \Omega_{ab} \rangle$  and  $\langle \sigma_a^c \rangle \langle \sigma_{bc} \rangle$ . For this reason, we call it the pre-effective Einstein equation rather than the effective Einstein equation. In order to make it meaningful, it is necessary to express  $\langle K_{ab} \rangle$  in terms of geometrical quantities in the bulk (i.e., the bulk metric and curvature) and the brane energy-momentum tensor. This problem is addressed in the next section.

### §3. The mean extrinsic curvature $\langle K_{ab} \rangle$

#### 3.1. Junction condition for $\langle K_{ab} \rangle$

The equation we solve is Eq. (2.19), which we present below for convenience:

$$-\frac{1}{\kappa_{(n)}^2} [\mathcal{F}_{ab}] = 2\tau_a^c \langle K_{bc} \rangle - \frac{(n-3)\lambda - \tau}{n-2} \langle K_{ab} \rangle - \langle K \rangle \left( \tau_{ab} + \frac{\lambda - \tau}{n-2} \gamma_{ab} \right). \quad (3.1)$$

The task is to solve this equation for  $\langle K_{ab} \rangle$ . We call this the second Gauss equation. Battye et al.<sup>12)</sup> solved this equation in two particular cases. One is the case in which the elements of the brane energy-momentum tensor,  $\tau_{ab}$ , are sufficiently smaller than the brane tension,  $\lambda$ , so that the low energy expansion is valid. The other is the case

in which the brane energy-momentum tensor takes the perfect fluid form. Here we seek a general solution without particular assumptions concerning the brane energy-momentum tensor.

For convenience, we introduce the “hatted” energy-momentum tensor,  $\hat{\tau}_{ab}$ , which differs from the original one by a trace term :

$$\hat{\tau}_{ab} \equiv \tau_{ab} - \frac{(n-3)\lambda - \tau}{2(n-2)}\gamma_{ab} = \bar{T}_{ab} - \frac{1}{2(n-2)}\bar{T}\gamma_{ab}. \quad (3.2)$$

Using this, the junction condition (2.18) becomes

$$[K_{ab}] = \kappa_{(n)}^2 \left( -\hat{\tau}_{ab} + \frac{\hat{\tau}}{n-3}\gamma_{ab} \right), \quad (3.3)$$

and the second Gauss equation, (3.1), becomes

$$-[\mathcal{F}_{ab}] = 2\kappa_{(n)}^2 \hat{\tau}_{(a}{}^c \langle K_{b)c} \rangle + \langle K \rangle [K_{ab}]. \quad (3.4)$$

Below we solve Eq. (3.4) to obtain the mean extrinsic curvature. Our method consists of two parts. First, we obtain the trace of the mean extrinsic curvature  $\langle K \rangle$  by introducing the inverse of the hatted tensor  $\hat{\tau}_{ab}$ . Second, with  $\langle K \rangle$  known, we rewrite the second Gauss equation as a matrix equation for  $\langle K_{ab} \rangle$ . This matrix equation can be solved by using the tetrad (more precisely, the vielbein) decomposition of  $\hat{\tau}_{ab}$ .

First, let us solve for the trace of the mean extrinsic curvature. We regard  $\hat{\tau}_{ab}$  as an  $(n-1) \times (n-1)$  symmetric matrix and introduce its inverse matrix,  $(\hat{\tau}^{-1})^{ab}$ , assuming  $\det(\hat{\tau}_{ab}) \neq 0$ . By definition, the inverse matrix  $(\hat{\tau}^{-1})^{ab}$  satisfies

$$(\hat{\tau}^{-1})^{ac} \hat{\tau}_{cb} = \delta^a_b. \quad (3.5)$$

We multiply the second Gauss equation (3.4) by this inverse matrix  $(\hat{\tau}^{-1})^{ab}$  and take the trace to obtain

$$\kappa_{(n)}^2 \langle K \rangle = \frac{(n-3)(\hat{\tau}^{-1})^{ab}[\mathcal{F}_{ab}]}{(n-3)^2 - \hat{\tau}^m{}_m (\hat{\tau}^{-1})^n{}_n}, \quad (3.6)$$

where Eq. (3.3) has been used. We note that the denominator is zero if  $\hat{\tau}^a{}_a (\hat{\tau}^{-1})^b{}_b = (n-3)^2$ , and thus a special consideration is needed in this case. We defer this to a future study.

Inserting the solution (3.6) into the second Gauss equation (3.4), we obtain the matrix equation

$$2\kappa_{(n)}^2 \hat{\tau}_{(a}{}^c \langle K_{b)c} \rangle = -[\tilde{\mathcal{F}}_{ab}], \quad (3.7)$$

where  $[\tilde{\mathcal{F}}_{ab}]$  is defined by

$$[\tilde{\mathcal{F}}_{ab}] \equiv [\mathcal{F}_{ab}] + \langle K \rangle [K_{ab}] = [\mathcal{F}_{ab}] + \frac{(n-3)(\hat{\tau}^{-1})^{cd}[\mathcal{F}_{cd}]}{(n-3)^2 - \hat{\tau}^m{}_m (\hat{\tau}^{-1})^n{}_n} \left( -\hat{\tau}_{ab} + \frac{\hat{\tau}}{n-3}\gamma_{ab} \right). \quad (3.8)$$

To solve Eq. (3.7), we use the vielbein decomposition. First, we introduce a set of orthonormal basis vectors  $e_a^{(i)}$  ( $i = 0, 1, \dots, n-2$ ), where  $e_a^{(0)}$  is timelike and the rest are spacelike. They satisfy the orthonormality conditions

$$e_{(i)}^a e_{a(j)} = \eta_{(i)(j)}, \quad (3.9)$$

$$\sum_{i,j=0}^{n-2} \eta_{(i)(j)} e_a^{(i)} e_b^{(j)} = \sum_{j=0}^{n-2} e_a^{(j)} e_{b(j)} = \gamma_{ab}, \quad (3.10)$$

where  $\eta_{(i)(j)}$  is the Minkowski metric. We note that here we do not adopt the standard Einstein convention for the summation over the frame indices ( $i$ ); instead, we explicitly insert the symbol  $\sum$  when the summation is to be taken.

Because the matrix  $\hat{\tau}_{ab}$  is symmetric, it can be diagonalized by an orthogonal matrix  $P_{ab}$ :

$$(P^{-1})_a{}^c \hat{\tau}_{cd} P^d{}_b = \sum_i \hat{\tau}_{(i)} e_a^{(i)} e_{b(i)}. \quad (3.11)$$

We note that  $P_{ab}$  represents a local Lorentz transformation of the frame. Thus the frame determined by  $\bar{e}_a^{(i)} \equiv P_a{}^c e_c^{(i)}$  corresponds to a local Lorentz frame in which the hatted tensor  $\hat{\tau}_{ab}$  (and hence the original energy-momentum tensor  $\tau_{ab}$ ) is diagonalized:

$$\hat{\tau}_{ab} = \sum_i \hat{\tau}_{(i)} \bar{e}_a^{(i)} \bar{e}_{b(i)}. \quad (3.12)$$

Inserting this into the matrix equation (3.7) and contract in it with  $\bar{e}_{(i)}^a \bar{e}_{(j)}^b$ , we obtain

$$\kappa_{(n)}^2 (\hat{\tau}_{(i)} + \hat{\tau}_{(j)}) \bar{e}_{(i)}^a \bar{e}_{(j)}^b \langle K_{ab} \rangle = -\bar{e}_{(i)}^a \bar{e}_{(j)}^b [\tilde{\mathcal{F}}_{ab}], \quad (3.13)$$

or equivalently,

$$\kappa_{(n)}^2 (\hat{\tau}_{(i)} + \hat{\tau}_{(j)}) \langle K_{(i)(j)} \rangle = -[\tilde{\mathcal{F}}_{(i)(j)}], \quad (3.14)$$

where  $K_{(i)(j)}$  and  $F_{(i)(j)}$  denote the vielbein components. We recall that this equation holds for each pair of ( $i$ ) and ( $j$ ). Then, we divide both sides by  $\hat{\tau}_{(i)} + \hat{\tau}_{(j)}$ , assuming this is non-zero, and obtain

$$\kappa_{(n)}^2 \langle K_{ab} \rangle = - \sum_{i,j} \frac{1}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} \bar{e}_a^{(i)} \bar{e}_b^{(j)} [\tilde{\mathcal{F}}_{(i)(j)}]. \quad (3.15)$$

We also note that this result is valid only if we have  $\hat{\tau}_{(i)} + \hat{\tau}_{(j)} \neq 0$  for all possible pairs of ( $i$ ) and ( $j$ ). We need a special treatment in the case that any of the demonimators are zero. Again, we defer treatment this special case to a future study.

The diagonal term can be written

$$\sum_{i=j} \frac{1}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} \bar{e}_a^{(i)} \bar{e}_b^{(j)} [\tilde{\mathcal{F}}_{(i)(j)}] = \frac{1}{2} (\hat{\tau}^{-1})_a{}^c [\tilde{\mathcal{F}}_{cb}]. \quad (3.16)$$



Using this, we arrive at the final result,

$$\begin{aligned} \kappa_{(n)}^2 \langle K_{ab} \rangle = & -\frac{1}{2}(\hat{\tau}^{-1})_a{}^c [\mathcal{F}_{bc}] + \frac{(n-3)(\hat{\tau}^{-1})^{de} [\mathcal{F}_{de}]}{2((n-3)^2 - \hat{\tau}^m_m (\hat{\tau}^{-1})^n_n)} \left( \gamma_{ab} - \frac{\hat{\tau}^c_c}{n-3} (\hat{\tau}^{-1})_{ab} \right) \\ & - \sum_{i \neq j} \frac{1}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} \bar{e}_a^{(i)} \bar{e}_b^{(j)} [\omega_{(i)(j)}]. \end{aligned} \quad (3.17)$$

We refer to this as the junction condition for the mean of the extrinsic curvature, which is a counterpart to the conventional junction condition for the jump of the extrinsic curvature, Eq. (2.18). If convenient, the above result can be decomposed into its trace and traceless parts,

$$\kappa_{(n)}^2 \langle K \rangle = \frac{(n-3)(\hat{\tau}^{-1})^{ab} [\mathcal{F}_{ab}]}{(n-3)^2 - \hat{\tau}^m_m (\hat{\tau}^{-1})^n_n}, \quad (3.18)$$

$$\begin{aligned} \kappa_{(n)}^2 \langle \sigma_{ab} \rangle = & -\frac{1}{2}(\hat{\tau}^{-1})_a{}^c [\mathcal{F}_{bc}] \\ & + \frac{(\hat{\tau}^{-1})^{cd} [\mathcal{F}_{cd}]}{2\{(n-3)^2 - \hat{\tau}^m_m (\hat{\tau}^{-1})^n_n\}} \left( \frac{(n-3)^2}{n-1} \gamma_{ab} - \hat{\tau}^e_e (\hat{\tau}^{-1})_{ab} \right) \\ & - \sum_{i \neq j} \frac{1}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}} \bar{e}_a^{(i)} \bar{e}_b^{(j)} [\omega_{(i)(j)}]. \end{aligned} \quad (3.19)$$

To conclude this subsection, we have obtained the general solution for the mean extrinsic curvature expressed in terms of the bulk curvature and the brane energy-momentum. The result constitutes the junction condition for the mean of the extrinsic curvature across the brane, and it is analogous to the well-known junction condition for the jump of the extrinsic curvature. As we have noted, this result is valid in all but a few cases, in which the equation to be solved is singular. A careful analysis of these singular cases will be given in the future.

### 3.2. Effect of $\langle K_{ab} \rangle$ on the brane

In the previous subsection, we obtained the junction condition for the mean extrinsic curvature. In this subsection, we discuss its effect on the brane from several points of view.

#### 3.2.1. Effectively anisotropic fluid

In the frame of the basis vectors  $\bar{e}_a^{(i)} (= P_a{}^b e_b^{(i)})$ , we have the diagonalized energy-momentum tensor

$$\tau_{ab} = \sum \tau_{(i)} \bar{e}_a^{(i)} \bar{e}_{b(i)}, \quad (3.20)$$

where  $\tau_{(i)} = \tau_{ab} \bar{e}_{(i)}^a \bar{e}^{b(i)}$ , and the quadratic term  $S_{ab}$  is given by

$$\begin{aligned} S_{ab} \bar{e}_{(i)}^a \bar{e}_{(j)}^b = & \left( \frac{1}{n-3} \tau \tau_{(i)} - \frac{\tau^2}{2(n-3)} - \frac{n-3}{n-2} \tau_{(i)}^2 + \frac{n-2}{2(n-3)} \left( \sum_k \tau_{(k)}^2 \right) \right) \eta_{(i)(j)} \\ \equiv & S_{(i)} \eta_{(i)(j)}. \end{aligned} \quad (3.21)$$

With these, we can rewrite the effective Einstein equation in terms of the vielbein components.

The diagonal components of the Einstein equation are

$$\bar{G}_{(j)(j)} = -\bar{\Lambda} + \kappa_{(n-1)}^2 \tau_{(j)} + \kappa_{(n-1)}^2 S_{(j)} + \langle \Omega_{(j)(j)} \rangle + \sum_k \langle \sigma_{(j)}^{(k)} \rangle \langle \sigma_{(j)(k)} \rangle, \quad (3.22)$$

for each  $(j)$ , and the off-diagonal components are

$$\bar{G}_{(i)(j)} = \langle \Omega_{(i)(j)} \rangle + \sum_k \langle \sigma_{(i)}^{(k)} \rangle \langle \sigma_{(k)(j)} \rangle, \quad (3.23)$$

for  $(i) \neq (j)$ . The latter expression shows that the source terms for the off-diagonal components originate from  $\langle \Omega_{(i)(j)} \rangle$  and  $\langle \sigma_{(i)(j)} \rangle$ . This should be compared to the  $Z_2$  symmetric case, in which the off-diagonal components are due only to  $\mathcal{E}_{ab}$ .<sup>19)</sup> In the present case, in addition to  $\Omega_{ab}$ , which is a generalization of  $\mathcal{E}_{ab}$ , there is a contribution to off-diagonal components from the new term,  $\langle \sigma_a{}^c \rangle \langle \sigma_{cb} \rangle$ .

### 3.2.2. Low energy limit

We conjecture that the low energy regime, where  $|\tau_{ab}| \ll \lambda$ , Einstein gravity is recovered on the brane. For this reason, we believe that the contributions of the  $\omega_{ab}$  and  $S_{ab}$  terms become negligibly small. To examine this, let us consider the solution for the mean extrinsic curvature up to  $O(\tau_{ab}^2, \omega_{ab})$ . In this case, the inverse of the hatted energy-momentum tensor is given by

$$(\hat{\tau}^{-1})^{ab} = \frac{2(n-2)}{n-3} \lambda^{-1} \left( \gamma^{ab} + \frac{2(n-2)}{n-3} \lambda^{-1} \left( \tau^{ab} - \frac{\tau}{2(n-2)} \gamma^{ab} \right) + \dots \right). \quad (3.24)$$

Then  $\langle K_{ab} \rangle$  can be readily obtained as

$$\kappa_{(n)}^2 \langle K_{ab} \rangle = \frac{[\mathcal{F}]}{2(n-1)\lambda} \left\{ \gamma_{ab} + \frac{1}{\lambda} (\tau \gamma_{ab} - (n-2)\tau_{ab}) \right\} + \mathcal{O}(\tau_{ab}^2, \omega_{ab}). \quad (3.25)$$

Using this, the effective Einstein equation becomes

$$\bar{G}_{ab} = -\bar{\Lambda}^{LE} \gamma_{ab} + \kappa_{(n-1)}^{LE} \tau_{ab} + O(\tau_{ab}^2, \omega_{ab}), \quad (3.26)$$

where

$$\kappa_{(n-1)}^{LE} = \frac{n-3}{4(n-2)} \kappa_{(n)}^4 \lambda - \frac{(n-2)(n-3)[\mathcal{F}]^2}{4(n-1)^2 \kappa_{(n)}^4 \lambda^3}, \quad (3.27)$$

$$\bar{\Lambda}^{LE} = \frac{n-3}{2(n-1)} \langle \mathcal{F} \rangle + \frac{n-3}{8(n-2)} (\kappa_{(n)}^2 \lambda)^2 + \frac{(n-2)(n-3)[\mathcal{F}]^2}{8\kappa_{(n)}^4 (n-1)^2 \lambda^2}. \quad (3.28)$$

Thus, Einstein gravity is recovered. However, in contrast to our naive expectation, the contribution of the mean extrinsic curvature gives rise to new correction terms from the bulk, both to the gravitational constant and to the cosmological constant, which are not necessarily constant.

### 3.2.3. Perfect fluid case

We next consider the case of a perfect fluid, in which we have

$$\tau_{ab} = \rho u_a u_b + p(\gamma_{ab} + u_a u_b). \quad (3.29)$$

In this case, it is convenient to decompose  $\omega_{ab}$  as

$$\omega_{ab} = \kappa_{(n)}^2 \left\{ \rho^\omega u_a u_b + \frac{1}{n-2} \rho^\omega (\gamma_{ab} + u_a u_b) + 2q_{(a}^\omega u_{b)} + \pi_{ab}^\omega \right\}. \quad (3.30)$$

Then we obtain

$$\begin{aligned} \kappa_{(n)}^2 \langle K_{ab} \rangle = & -\frac{[\mathcal{F}]}{2(n-1)(\rho+\lambda)^2} \{ (\rho+p)u_a u_b + (\rho+\lambda)\gamma_{ab} \} \\ & + \frac{n-2}{(n-3)(\rho+\lambda)} \left\{ -\frac{p-\lambda}{\rho+\lambda} [\rho^\omega] u_a u_b + \frac{1}{n-2} [\rho^\omega] (\gamma_{ab} + u_a u_b) + 2[q_{(a}^\omega] u_{b)} \right\} \\ & - \frac{n-2}{\rho + (n-2)p - (n-3)\lambda} [\pi_{ab}^\omega]. \end{aligned} \quad (3.31)$$

This is consistent with the result of Battye et al.<sup>12)</sup>

## §4. Conclusion

In this paper, we considered a general codimension-1 brane in an arbitrary number of dimensions without  $Z_2$  symmetry and derived the junction conditions in a complete form. More specifically, we obtained expressions for both the jump and the mean of the extrinsic curvature in terms of the bulk curvature tensor and the brane energy-momentum tensor. With this result, we derived the effective Einstein equation on the brane in its most general form, which is a generalization of the Shiromizu-Maeda-Sasaki equation<sup>11)</sup> to the case in which  $Z_2$  symmetry does not exist.

The derived effective Einstein equation has a new term arising from the mean extrinsic curvature, and this new term leads to the appearance effectively anisotropic matter on the brane. This gives rise to the possibility of anisotropic brane worlds with an anisotropic bulk geometry, even in the case of an isotropic brane energy-momentum tensor.

Thus, our result is a basic equation for the hybrid brane world scenario, in which some spatial dimensions on the brane are Kaluza-Klein compactified.<sup>5)</sup> Also, it provides a basis for higher codimension brane worlds in which a higher codimension brane is regularized by a codimension-1 brane with extra dimensions on the brane compactified to an infinitesimally small size.

There is, however, one subtlety in our derivation. We had to assume the energy-momentum tensor on the brane to satisfy several conditions. When these conditions are not satisfied, the equation for the mean extrinsic curvature is singular. We plan to investigate this singular case in a future publication.

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### Appendix A

#### — Energy-Momentum Conservation —

Here, for reference, we evaluate the energy-momentum conservation law on the brane. From the Codazzi equation (2.4), we have

$$\bar{\nabla}_b K - \bar{\nabla}_a K^a_b = \mathcal{R}_{MN} n^M \gamma^N_b = \mathcal{G}_{MN} n^M \gamma^N_b. \quad (\text{A}\cdot 1)$$

Applying the junction condition (2.18) to the above, we obtain

$$\bar{\nabla}_a \tau^a_b = -[\mathcal{R}_{MN}] n^M \gamma^N_b, \quad (\text{A}\cdot 2)$$

$$\bar{\nabla}_b \langle K \rangle - \bar{\nabla}_a \langle K^a_b \rangle = \frac{n-2}{n-1} \bar{\nabla}_b \langle K \rangle - \bar{\nabla}_a \langle \sigma^a_b \rangle = \langle \mathcal{R}_{MN} \rangle n^M \gamma^N_b, \quad (\text{A}\cdot 3)$$

where  $\sigma$  is the traceless part of  $K_{ab}$ . The first equation reveals the violation of the energy-momentum conservation in the case that there is a jump in the bulk Ricci tensor. The second equation describes the exchange of the effective energy-momentum due to the mean extrinsic curvature and the bulk energy-momentum.

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